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or area the discussion of the limits involved in their evaluation must be postponed. The treatment of all problems in limits belongs essentially to the differential and integral calculus; and it might be far better to leave these difficult questions until the analytic means for adequately handling them have been developed. We shall, however, sketch in the ideas which come up in the proof that a tangent exists at any point  $M$  of a circle. First choose a point  $P$  on one side of  $M$  and draw the secant  $MP$ . Then let  $P$  approach  $M$  as a limit. It may be shown that as  $P$  approaches  $M$ , the secant  $MP$  turns always in one direction about the point  $P$ . This comes of the fact that the circle is a convex curve. But  $MP$  does not turn as far as the line formed by producing the radius through  $P$ . Hence, analogously to Theorem 4, there exists a limiting direction of the secant  $MP$ . We have therefore established the existence of a tangential direction on one side of the point  $M$ . By taking the point  $P$  on the other side we may likewise establish the existence of a tangential direction on that side. These two directions may be shown to be opposite directions along the same line through  $M$ ; and the proof of the existence of a tangent is then complete. With these suggestions we shall leave the problem to the reader.

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## ON COMPLETE SYMMETRIC FUNCTIONS.\*

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By DR. E. D. ROE, Jr., Professor of Mathematics in Syracuse University.

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### PART I. INTRODUCTION.

#### 1. DEFINITIONS, NOTATION, AND OBJECT OF THE PAPER.

Let  $\phi(a_1, a_2, \dots, a_n)$  be any rational function of the  $a$ 's. Let  $s = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ i_1 & i_2 & i_3 & \dots & i_n \end{pmatrix}$ ,  $i_1, i_2, \dots, i_n$  being some permutation of  $1, 2, 3, \dots, n$ ; then  $s$  is an operator which converts the index  $r$  into  $i_r$ ; also  $s$  applied to  $\phi$ , converts it into  $\phi_i$ , a function in which the  $a$ 's have been permuted. This is expressed by writing

$$(1) \qquad s\phi = \phi_i.$$

Of such operators  $s$  there are  $n!$ . Applying each one to  $\phi$ , we get  $\phi_1, \phi_2, \dots, \phi_{n!}$ . Let

$$(2) \qquad \Phi = \phi_1 + \phi_2 + \dots + \phi_{n!}.$$

Then  $\Phi$  is a symmetric function of the  $a$ 's. In particular cases it may happen that  $\phi_1, \phi_2, \dots, \phi_{n!}$ , are not all different, but it can be proved that each dis-

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\*A paper presented February 1, 1904, before The Mathematical Club of Syracuse University.

tinct  $\phi$  is repeated the same number,  $r$ , of times, that the number of distinct  $\phi$ 's is  $n!/r$ , and that

$$(3) \quad \Phi = r(\phi_1 + \phi_2 + \dots + \phi_{n!/r}).$$

We may observe that  $\Phi$  and a determinant  $\Delta$  of the  $n$ th order are related to each other by means of  $s$ . For, by applying the different values of  $s$  to the second set of indices of the principal diagonal term  $a_{11} a_{22} \dots a_{nn}$ , and also by simultaneously taking account of the number of inversions of order,  $i$ , among  $i_1, i_2, \dots, i_n$ , we get,

$$(4) \quad (-1)^is(a_{11} a_{12} \dots a_{nn}) = (-1)^{i_{a_1 i_1} a_2 i_2 \dots a_n i_n},$$

that is, we get every term of the determinant in question. Hence in general to every term of  $\Delta$  corresponds a term of  $\Phi$ , and conversely, while in particular cases [as in (3)] to  $r$  terms of  $\Delta$  may correspond the same term of  $\Phi$ .

We limit the present discussion to integral symmetric functions. It can be proved that integral symmetric functions can be reduced to sums of functions, each of which is homogeneous by itself, and of the type

$$(5) \quad \Sigma a_1^{p_1} a_2^{p_2} \dots a_n^{p_n}, \text{ with } p_1 \geq p_2 \geq \dots \geq p_n \geq 0.$$

This latter type is called a monomial symmetric function. It is often denoted by the symbol  $(p_1 p_2 \dots p_n)$ . The number  $w = p_1 + p_2 + \dots + p_n$  is defined as the weight and the number  $p_1$  as the order of the function. The simplest of the monomial symmetric functions are the elementary functions

$$(6) \quad \Sigma a_1, \Sigma a_1 a_2, \Sigma a_1 a_2 a_3, \dots$$

and the sums of the powers

$$(7) \quad \Sigma a_1^r = s_r = (r).$$

A complete symmetric function of weight  $w$  is defined as the sum of all the monomial symmetric functions of weight  $w$ . It is here denoted by  $t_w$ .\*

We have

$$(8) \quad t_w = \Sigma \Sigma a_1^{p_1} a_2^{p_2} \dots a_n^{p_n}.$$

For example,

$$(9) \quad t_4 = \Sigma a_1^4 + \Sigma a_1^3 a_2 + \Sigma a_1^2 a_2^2 + \Sigma a_1^2 a_2 a_3 + \Sigma a_1 a_2 a_3 a_4.$$

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\*Other notations for  $t_w$  are  $H(a_1, a_2, \dots, a_n)$  and  $nH_w$  or  $II_w$ . Wronski designates  $H(a_1, a_2, \dots, a_n)$  as the "Alephfunktion" of  $a_1, a_2, \dots, a_n$ . C. Smith (Treatise on Algebra, p. 239; §250) uses  $nH_w$ , not to designate the function, but the number of terms contained in it. Burnside and Panton, Theory of Equations, 2nd Edition, p. 297, use  $II_w$  for  $t_w$ , and the last three authors refer to the function as "homogeneous products." See further "Encyklopaedie der Mathematischen Wissenschaften," Teil I, Band I, Heft 4, p. 465.

We shall farther denote the elementary functions by the  $b$ 's, where  $b_r = a_r/a_0$ , and then

$$(10) \quad \Sigma a_1 a_2 \dots a_r = (-1)^r b_r,$$

$$(11) \quad a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0, \text{ or } x^n + b_1 x^{n-1} + \dots + b_n = 0,$$

is the equation which has the  $a$ 's for roots. Also

$$(12) \quad a_0^\pi \Sigma a_1^{p_1} a_2^{p_2} \dots a_n^{p_n} = \Sigma A_\kappa a_0^{\kappa_0} a_1^{\kappa_1} \dots a_n^{\kappa_n} \text{ (the expressibility proposition),}$$

$$(13) \quad \kappa_1 + 2\kappa_2 + \dots + n\kappa_n = w, \text{ (the weight proposition),}$$

$$(14) \quad \kappa_0 + \kappa_1 + \dots + \kappa_n = \pi, \pi = p_1, \text{ (the order proposition).}$$

The  $t$ 's are analogous to the  $b$ 's (or elementary symmetric functions) in many of their properties,\* while in other relations they play a rôle analogous to that of the  $s$ 's.

The object of this paper is to state and solve some problems in symmetric functions, to sketch the theory of the  $t$ 's, and to exhibit tables, which express up to weight five, homogeneous products of the  $t$ 's in terms of homogeneous products of other symmetric functions, and vice versa, and to point out certain properties of such tables.

## 2. THE STATEMENT OF SOME PROBLEMS IN SYMMETRIC FUNCTIONS.

### (i). *On Expressibility. Fundamental Systems.*

Besides the problem on expressibility whose solution is stated in (12), others have arisen. This leads to the concept of a fundamental system. A fundamental system for symmetric functions is any system of rational symmetric functions, by means of which any rational symmetric function can be rationally expressed.† Of such systems the following are known to exist:

$$(15) \quad \begin{array}{ll} b_1, b_2, \dots, b_n; \ddagger & s_1, s_2, \dots, s_n; \S \\ s_1, s_3, \dots, s_{2n-1}; \parallel & s_{v_1}, s_{v_2}, \dots, s_{v_n}. \P \end{array}$$

In the last system the  $v$ 's are the first  $n$  integers which are not multiples

\*This appears to have its basis in the relations expressed in (35) and (36) of this paper.

†See Ency. der Math. Wiss. loc. cit., p. 455.

‡Contained in (12).

§It can be proved that

$$(p_1 p_2 \dots p_n) = \begin{vmatrix} u_{11} u_{12} \dots u_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ v_{n1} v_{n2} \dots v_{nn} \end{vmatrix}, \text{ where } u_{11} = s_{p_1}, u_{22} = s_{p_2}, u_{12} u_{21} = s_{p_1+p_2},$$

$$u_{12} u_{23} u_{31} = s_{p_1+p_2+p_3}, \text{ etc.}$$

This proves  $s_1, s_2, \dots, s_n$  a fundamental system. See Am. Math. Monthly, Vol. 5, p. 161.

¶Proved by Borchardt. Gesammelte Werke, p. 107.

‡Proved by Vahlen. Ueber Fundamentalsysteme fuer Symmetrische Functionen. Acta Mathematica. Band 23, p. 91.

of a given integer, and this system is the generalization of and contains the two preceding systems. It is evident from (33) of this paper that

$$(16) \quad t_1, t_2, \dots, t_n; \quad t_1, t_2, t_3, \dots, t_{2n-1}; \quad t_1, t_2, t_3, \dots, t_{2n};$$

are also fundamental systems, but it would be interesting to know if the last two would remain fundamental systems, if such of the  $t$ 's were omitted as would make these systems analogues of the last two systems of (15). This question is left for future investigation.

(ii). *Other Problems.*

It is well known that the general equation (10) can not be solved algebraically when  $n > 4$ , but in case the roots enter an expression symmetrically, it is not necessary to find them in order to find the value of the expression. Thus symmetric functions solve problems where all the roots enter symmetrically without the necessity of finding them:

a. To find the general term in the development of

$$(17) \quad \phi(x) = \frac{d_0 + d_1x + \dots + d_nx^n}{a_0 + a_1x + \dots + a_nx^n} = \frac{c_1 + 2c_2x + 3c_3x^2 + \dots + (m+1)c_{m+1}x^m}{1 + b_1x + b_2x^2 + \dots + b_nx^n}.$$

b. To find the value of

$$(18) \quad \sum_{\kappa=0}^{\kappa=\infty} \frac{\kappa r}{\kappa!}.$$

The solutions of these problems are given later (Part III).

## PART II. THE THEORY OF THE $t$ 'S.

We have

$$(19) \quad x^n + b_1x^{n-1} + b_2x^{n-2} + \dots + b_n = (x - a_1)(x - a_2) \dots (x - a_n),$$

$$(20) \quad 1 + b_1x + b_2x^2 + \dots + b_nx^n = (1 - a_1x)(1 - a_2x) \dots (1 - a_nx), \text{ whence}$$

$$(21) \quad \frac{1}{1 + b_1x + b_2x^2 + \dots + b_nx^n} = t_0 + t_1x + t_2x^2 + \dots,^* \text{ or}$$

$$(22) \quad 1 = (t_0 + t_1x + t_2x^2 + \dots)(1 + b_1x + b_2x^2 + \dots + b_nx^n).$$

In (22) the coefficient of  $x^r = 0$ , if  $r > 0$ , hence

$$(23) \quad a_0t_r + a_1t_{r-1} + \dots + a_rt_0 = 0.$$

From the symmetry of (23) it follows that  $t_r$  is the same function of the  $a$ 's that  $a_r$  is of the  $t$ 's. Changing  $r$  into  $r-1$ , etc., in (23) we obtain  $r$  equations. Solving them for  $t_r$ , and noticing that  $t_0 = 1$ , we have,

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\*See Burnside and Panton, l. c., p. 297.

$$(24) \quad a_0{}^r t_r = (-1)^r \begin{vmatrix} a_1 a_2 & \dots & a_r \\ a_0 a_1 & \dots & a_{r-1} \\ 0 & a_0 & \dots & a_{r-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 a_1 \end{vmatrix} = (-1)^r [1 \ 2 \dots r]_a.$$

Also by the preceding remark,

$$(25) \quad t_0{}^r a_r = (-1)^r [1 \ 2 \dots r]_t.$$

Writing (21) in the form  $(1+\theta)^{-1} = t_0 + t_1 x + \dots$ , where  $\theta = b_1 x + \dots + b_n x^n$ , and developing by the binomial and multinomial theorems, and equating the coefficients of  $x^r$ ,

$$(26) \quad a_0{}^r t_r = \sum (-1)^{\kappa_1 + \kappa_2 + \dots + \kappa_r} \frac{(\kappa_1 + \kappa_2 + \dots + \kappa_r)!}{\kappa_1! \kappa_2! \dots \kappa_r!} a_0^{\kappa_0} a_1^{\kappa_1} \dots a_r^{\kappa_r}.$$

Similarly,

$$(27) \quad t_0{}^r a_r = \sum (-1)^{\kappa_1 + \kappa_2 + \dots + \kappa_r} \frac{(\kappa_1 + \kappa_2 + \dots + \kappa_r)!}{\kappa_1! \kappa_2! \dots \kappa_r!} t_0^{\kappa_0} t_1^{\kappa_1} \dots t_r^{\kappa_r},$$

$$\kappa_0 + \kappa_1 + \dots + \kappa_r = \kappa_1 + 2\kappa_2 + \dots + r\kappa_r = r.$$

By equating (26) and (24), we get the development of the determinant in the right member of (24). If in (24)  $a_0 = a_1 = \dots = a_r = 1$ ,  $0 < r < n+1$ , the determinant vanishes, and

$$(28) \quad \sum (-1)^{\kappa_1 + \kappa_2 + \dots + \kappa_n} \frac{(\kappa_1 + \kappa_2 + \dots + \kappa_r)!}{\kappa_1! \kappa_2! \dots \kappa_r!} = 0,$$

i. e., the sum of the coefficients in the development of  $t_r$  expressed in terms of the  $a$ 's vanishes. From (20) we get

$$(29) \quad \log(1 + b_1 x + \dots + b_n x^n) = - \sum \frac{s_r x^r}{r} = f_1(a)x + f_2(a)x^2 + \dots + f_r(a)x^r + \dots$$

$$(30) \quad -\log(1 + t_1 x + \dots) = - \sum \frac{s_r x^r}{r} = -f_1(t)x - f_2(t)x^2 + \dots + f_r(t)x^r + \dots,$$

whence

$$(31) \quad s_r = -r f_r(a) = (-1)^r \begin{vmatrix} b_1 & 1 & 0 & \dots & 0 \\ 2b_2 & b_1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & b_r & b_{r-1} & \dots & b_1 \end{vmatrix}$$

$$= r \sum (-1)^{\kappa_1 + \dots + \kappa_r} \frac{(\kappa_1 + \dots + \kappa_r - 1)!}{\kappa_1! \kappa_2! \dots \kappa_r!} b_1^{\kappa_1} b_2^{\kappa_2} \dots b_r^{\kappa_r}, *$$

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\*Burnside and Panton, l. c., p. 298.

where with  $0 < r < n+1$ , the sum of the coefficients of  $s_r$  (in terms of  $b$ 's)  $= -1$ , and

$$(32) \quad \sum (-1)^{\kappa_1 + \dots + \kappa_r} \frac{(\kappa_1 + \dots + \kappa_r - 1)!}{\kappa_1! \dots \kappa_r!} = -\frac{1}{r}.$$

Similarly,

$$(33) \quad s_r = (-1)^{r+1} \begin{vmatrix} t_1 & 1 & 0 & \dots & 0 \\ 2t_2 & t_1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & t_r & t_{r-1} & \dots & t_1 \end{vmatrix}$$

$$= -r \sum (-1)^{\kappa_1 + \dots + \kappa_r} \frac{(\kappa_1 + \dots + \kappa_r - 1)!}{\kappa_1! \dots \kappa_r!} t_1^{\kappa_1} \dots t_r^{\kappa_r}, \text{ with}$$

$$(34) \quad \text{the sum of the coefficients of } s_r \text{ (in terms of the } t\text{'s)} = 1.$$

We may also write (29) and (30) in the forms

$$(35) \quad 1 + b_1 x + \dots = e^{-\sum (s_r x^r / r)},$$

$$(36) \quad 1 + t_1 x + \dots = e^{\sum (s_r x^r / r)},$$

and obtain,

$$(37) \quad b_r = \sum (-1)^{\kappa_1 + \dots + \kappa_r} \frac{(\kappa_1 + \dots + \kappa_r - 1)!}{\kappa_1! \dots \kappa_r!} \left(\frac{s_1}{1}\right)^{\kappa_1} \left(\frac{s_2}{2}\right)^{\kappa_2} \dots \left(\frac{s_r}{r}\right)^{\kappa_r}, *$$

$$(38) \quad t_r = \sum \frac{1}{\kappa_1! \dots \kappa_r!} \left(\frac{s_1}{1}\right)^{\kappa_1} \left(\frac{s_2}{2}\right)^{\kappa_2} \dots \left(\frac{s_r}{r}\right)^{\kappa_r}.$$

In fact (35) and (36) show that changing  $s$  into  $-s$ , converts  $b$  into  $t$ . This principle enables us to derive many new relations instantly from given relations. Thus (38) is derived in this way from (37), and the Newtonian identities

$$(39) \quad s_r + s_{r-1}b_1 + s_{r-2}b_2 + \dots + rb_r = 0, \dagger \quad r=1, 2, \dots$$

yield by this operation at once the analogous relations,

$$(40) \quad s_r + s_{r-1}t_1 + s_{r-2}t_2 + \dots + rt_r = 0. \ddagger$$

The principle also yield's this general theorem:

*Any symmetric function expressed in terms of  $s$ 's may be expressed in terms of  $t$ 's, by changing  $s$  into  $-s$ , and then changing  $b$  into  $t$ , in each  $s$  expressed in terms of  $b$ 's.*

\*Burnside and Panton, l. c., p. 298.

†Burnside and Panton, l. c., p. 290.

‡This relation has also been observed by L. Crocchi, "Una relazione fra a funzioni simmetriche semplici e le funzioni simmetriche complete." G. Battaglini's "Giornale Mathematiche," etc., XVIII, 377-380.

The formulas (40) give also

$$(41) \quad t_r = \frac{(-1)^{r+1}}{r!} \begin{vmatrix} s_1 & 1 & 0 & 0 & \dots & 0 \\ s_2 & -s_1 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_r & -s_{r-1} & \dots & \dots & \dots & -s_1 \end{vmatrix},$$

where the sum of the coefficients is 1, or by (38),

$$\sum \frac{1}{\kappa_1! \dots \kappa_r!} \left(\frac{1}{1}\right)^{\kappa_1} \left(\frac{1}{2}\right)^{\kappa_2} \dots \left(\frac{1}{r}\right)^{\kappa_r} = 1.$$

And the formulas (39) give

$$(42) \quad b_r = \frac{(-1)^r}{r!} \begin{vmatrix} s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ s_r & s_{r-1} & \dots & \dots & s_1 \end{vmatrix},$$

where by (37),

$$(43) \quad \sum (-1)^{\kappa_1 + \dots + \kappa_r} \frac{1}{\kappa_1! \dots \kappa_r!} \left(\frac{1}{1}\right)^{\kappa_1} \left(\frac{1}{2}\right)^{\kappa_2} \dots \left(\frac{1}{r}\right)^{\kappa_r} = 0.$$

Some other relations satisfied by the  $t$ 's are the following, which are stated without details of proof:

$$(44) \quad s_r = -(b_1 t_{r-1} + 2b_2 t_{r-2} + 3b_3 t_{r-3} + \dots + r b_r t_0).$$

This is obtained by expanding (31) in terms of the elements of the first column, and having regard to (24), or by differentiating (29) with respect to  $x$ , and using (21).

$$(45) \quad \frac{\partial s_r}{\partial b_\kappa} = -r t_{r-\kappa}, \quad \frac{\partial s_r}{\partial t_\kappa} = r b_{r-\kappa}.$$

The first one of these can be obtained by differentiating (31) partially with respect to  $b_\kappa$ , and observing (26); the second by the remark following (38).

$$(46) \quad \frac{\partial t_{2r+\kappa}}{\partial b_\kappa} = -(t_r^2 + 2t_{2r} + 2t_1 t_{2r-1} + \dots + 2t_{r-1} t_{r+1}),$$

$$(47) \quad \frac{\partial t_{2r+1+\kappa}}{\partial b_\kappa} = -(2t_{2r+1} + 2t_1 t_{2r} + 2t_2 t_{2r-1} + \dots + 2t_r t_{r+1}).$$

(46) and (47) come from differentiating (21) partially with respect to  $b_\kappa$ . In (46) and (47) we may interchange  $t$  and  $b$ .

$$(48) \quad r\kappa \frac{\partial b_\kappa}{\partial s_r} + \frac{\partial s_\kappa}{\partial t_r} = 0, \quad r\kappa \frac{\partial t_\kappa}{\partial s_r} + \frac{\partial s_\kappa}{\partial b_r} = 0.$$

The first formula in (48) comes from combining (45) with the relation  $\frac{\partial b_\kappa}{\partial s_r} = -\frac{1}{r} b_{\kappa-r}$ ;\* the second by changing  $s$  into  $-s$ , and interchanging  $b$  and  $t$ .

$$\text{If } -\delta = \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2} + \dots + \frac{\partial}{\partial a_n},$$

$$(49) \quad \partial t_\kappa = (n + \kappa - 1) t_{\kappa-1}. \dagger$$

(49) can be easily proved by mathematical induction.

$$(50) \quad \frac{-(b_1 + b_3 x^2 + b_5 x^4 + \dots)}{(1 - a_1^2 x^2)(1 - a_2^2 x^2) \dots (1 - a_n^2 x^2)} = t_1 + t_3 x^2 + t_5 x^4 + \dots$$

(50) comes from using  $x$  and  $-x$  successively in (21).

$$(51) \quad H(a_1, a_2, \dots, a_n) = H_m(a_1, a_2, \dots, a_\kappa) + H_{m-1}(a_1, \dots, a_\kappa) \\ H(a_{\kappa+1}, \dots, a_n) + \dots H(a_{\kappa+1}, \dots, a_n). \ddagger$$

$$(52) \quad \frac{|a_1^{\kappa_1} a_2^{\kappa_2} \dots a_n^{\kappa_n}|}{|a_1^0 a_2^1 \dots a_n^{n-1}|} = \left| \begin{array}{cccc} t_{\kappa_1} & t_{\kappa_2} & \dots & t_{\kappa_n} \\ t_{\kappa_1-1} & t_{\kappa_2-1} & \dots & t_{\kappa_n-1} \\ \dots & \dots & \dots & \dots \\ t_{\kappa_1-n+1} & t_{\kappa_2-n+1} & \dots & t_{\kappa_n-n+1} \end{array} \right|. \S$$

$$(53) \quad \Pi_r = \sum \frac{a_1^{n+r-1}}{f'(a_1)}. \parallel$$

\*Burnside and Panton, l. c., p. 304. From this comes the relation

$$\frac{\partial t_\kappa}{\partial s_r} = \frac{1}{r} t_{\kappa-r}.$$

†This relation was communicated to me by Professor Gordan.

‡Ency. der Math. Wiss., l. c., p. 465.

§l. c., and for a proof of this theorem see Muir's Determinants, p. 170, §125.

||Burnside and Panton, l. c., p. 297. See also Ency. der Math. Wiss., l. c., p. 459.